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Non-linear solutions for smectic C liquid crystals in wedge and cylinder geometries

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This paper discusses some non-linear problems for smectic C liquid crystals based on the continuum theory proposed by Leslie *et al.* New restrictions on the nine elastic constants are also derived. Attention is restricted to samples involving concentric cylindrical layers in which both the layer thickness and the tilt angle are assumed to be constant. Non-linear solutions are presented for a sample contained in a wedge with an electric field applied across the bounding plates, extending earlier work by Carlsson *et al.*, and for a sample between two coaxial concentric circular cylinders to which an azimuthal magnetic field is applied. Fréedericksz thresholds, which may lead to the experimental determination of some of the elastic constants, are deduced. In the absence of an applied field it is found that, under suitable restrictions on the elastic constants, there is a critical wedge angle (or critical radius ratio in the concentric cylinder case) above which a variable non-linear symmetric solution satisfying the zero boundary conditions is energetically more favourable than the zero solution.

1. Introduction

The Fréedericksz transition caused by an electric field for cylindrical layers of smectic C liquid crystals confined in a wedge has recently been examined theoretically by Carlsson *et al.* [1] using the continuum theory proposed by Leslie *et al.* [2, 3]. This Fréedericksz transition partly depends on the elastic constants connected with layer deformations (the A_i constants mentioned below). These layer constants are normally difficult to obtain and the geometrical arrangement of the wedge problem provides a possible experimental design where the Fréedericksz type transition threshold can be measured.

As Carlsson *et al.* [1] were principally concerned with deriving the relationship between the critical voltage and the various material parameters, the results derived in [1] were obtained by linearizing the resulting Euler–Lagrange equation for the problem, firstly with respect to the smectic tilt angle θ and secondly with respect to the *c*-director phase angle ϕ which was assumed to depend only upon the angle α of the cylindrical polar coordinate system (r, α, z). One purpose of this article is to extend the analysis in [1], retaining the non-linearities in θ and ϕ, and derive a non-linear α-dependent solution to the wedge problem, the details of which are also of interest. This solution exists above a critical voltage and its existence depends upon additional inequalities being imposed which involve the elastic constants. A full non-linear energy comparison demonstrates that, under certain restrictions, the distorted configuration is energetically more favourable than the original orientation pattern. Thus one anticipates that the distorted solution occurs in preference to the uniform alignment and a Fréedericksz transition results. It is found that the non-linearly derived Fréedericksz threshold coincides with that found in [1] when it is linearized. The new relationship between the critical voltage, the tilt angle, the wedge angle and the elastic constants may provide a possible method for determining experimentally the elastic constants.

The plan of this article is as follows: in §2 the mathematical description of smectic C liquid crystals is outlined briefly. A new set of inequalities which the elastic constants must satisfy are derived in §3 together with the basic Euler–Lagrange equation. The full analysis for the α-dependent solution is presented in §4. In order to justify the assumption $\phi = \phi(\alpha)$ for the wedge problem for large samples, in §5 a linear solution $\phi = \phi(r, \alpha)$ is examined. In §6 the full analysis for a non-linear *r*-dependent solution is given. Here the sample of liquid crystal is confined at rest between two coaxial, concentric circular cylinders. A magnetic field is applied in the azimuthal direction so that it is everywhere tangential to circles perpendicular to the common axis

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and has its magnitude inversely proportional to the distance r from this axis. In this case the solution exists above a critical field strength which depends upon the tilt angle, elastic constants and the ratio of the cylinder radii. Above this strength, the distorted configuration is energetically more favourable compared with the initial orientation pattern. As in the wedge, one anticipates that above the critical strength the distorted solution occurs in preference to the uniform alignment and a Fréedericksz transition results. The relationship between the critical field strength, the geometry and the elastic constants may provide the experimentalist with a further method for determining some of the elastic constants. However, care may be necessary as there is a critical wedge angle, or critical radius ratio in the coaxial cylinder case, and it is only below these values that a Fréedericksz transition may be expected. In the absence of applied fields, under suitable restrictions on the A_i-constants, above these critical values a variable solution exists which is energetically more favourable than the initial pattern. The article concludes with a discussion in § 7.

2. Smectic C wedge and cylinder problems

Liquid crystals consist of elongated molecules where the molecular long axes locally align along a common direction in space which is generally denoted by the unit vector $\mathbf{n}$, called the director. Smectic C liquid crystals are known to form equidistant parallel layers in which $\mathbf{n}$ makes an angle θ with respect to the layer normal. Here the tilt angle θ is taken to be constant and the layers are assumed to be of constant thickness. Following de Gennes and Prost [4], a smectic C liquid crystal can be described by introducing the unit layer normal $\mathbf{a}$ and a unit vector $\mathbf{c}$ which is the unit orthogonal projection of the director $\mathbf{n}$ onto the smectic planes. The vectors $\mathbf{a}$ and $\mathbf{c}$ are subject to the constraints

$$\mathbf{a}\cdot\mathbf{a}=\mathbf{c}\cdot\mathbf{c}=1, \qquad \mathbf{a}\cdot\mathbf{c}=0 \tag{1}$$

since $\mathbf{a}$ and $\mathbf{c}$ are clearly unit and orthogonal. Since we are only concerned with samples which do not contain dislocations the relation

$$\nabla\times\mathbf{a}=\mathbf{0} \tag{2}$$

must also hold [5, 6]. The constraint (2) is known to restrict the possible number of equidistant layer structures and force the liquid crystals to form undistorted parallel layers which form planes, concentric cylinders, spheres or parts thereof. More complicated equidistant layerings consist of concentric circular tori, Dupin cyclides [4, 7–9] or parabolic cyclides [10–12].

It is mathematically convenient to introduce the unit vector $\mathbf{b}$ defined by

$$\mathbf{b}=\mathbf{a}\times\mathbf{c}. \tag{3}$$

If the layering structure is assumed to remain intact so that $\mathbf{a}$ remains fixed then knowledge of the c-director completely describes the alignment of the director $\mathbf{n}$ throughout the sample. For the solutions considered here the smectic planes form concentric cylinders whose common axis coincides with the z-axis of a system of cylindrical polar coordinates (r, α, z). We introduce the following ansatz for $\mathbf{a}, \mathbf{b}, \mathbf{c}$

$$\mathbf{a}=\hat{\mathbf{r}}, \tag{4}$$

$$\mathbf{b}=-\hat{\boldsymbol{\alpha}}\cos\phi+\hat{\mathbf{z}}\sin\phi, \tag{5}$$

$$\mathbf{c}=\hat{\boldsymbol{\alpha}}\sin\phi+\hat{\mathbf{z}}\cos\phi, \tag{6}$$

where we assume $\phi=\phi(r,\alpha)$ and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\mathbf{z}}$ are the unit basis vectors: r measures the outward radial distance, α is the usual polar angle and the z axis coincides with the apex of the wedge (the common axis of the cylinders). From the geometry the director $\mathbf{n}$ is simply

$$\begin{aligned}\mathbf{n}&=\mathbf{a}\cos\theta+\mathbf{c}\sin\theta\\ &=\hat{\mathbf{r}}\cos\theta+\hat{\boldsymbol{\alpha}}\sin\theta\sin\phi+\hat{\mathbf{z}}\sin\theta\cos\phi.\end{aligned} \tag{7}$$

As discussed in [1], for a wedge formed by two glass plates at an angle β the boundary conditions on the director allow us to set $\phi=0$ when $\alpha=0, \beta$. The electric field $\mathbf{E}$ follows the plane of the layers and is applied between the bounding plates at $\alpha=0, \beta$. This field is achieved when

$$\mathbf{E}=\frac{U}{r\beta}\hat{\boldsymbol{\alpha}}, \tag{8}$$

where U is the applied voltage across the cell. For fuller details of the wedge geometry the reader is referred to [1].

For the sample between concentric cylinders of radii a and b $(>a)$ the appropriate boundary conditions are $\phi=0$ when $r=a, b$. The magnetic field takes the form

$$\mathbf{B}=\frac{Ba}{r}\hat{\boldsymbol{\alpha}}, \tag{9}$$

where B is the magnetic field strength at $r=a$. This field may be achieved by passing an electric current along a wire situated along the z-axis.

It is known that when the strength of an electric (or magnetic) field increases as it is applied across a sample of liquid crystal there is a critical threshold, the Fréedericksz threshold, above which the director $\mathbf{n}$ begins to reorient itself as it is attracted or repelled by the field. Fréedericksz transitions in planar layers of smectic C have been discussed by Rapini [13] and, as mentioned above, transitions for cylindrical layers in a wedge have been investigated by Carlsson *et al.* [1]. A full analysis of Fréedericksz transitions for spherical layers of smectic C has been carried out by Atkin and Stewart

[14] for the usual cone and plate geometry. Fréedericksz transitions in cylindrical and spherical geometries for nematic liquid crystals have been examined by Atkin and Barratt [15].

3. Energies and the Euler–Lagrange equation

The bulk elastic energy w_b for a non-chiral smectic C liquid crystal in terms of **b** and **c** is [1, 3]

$$\begin{aligned} w_b = & \frac{1}{2}A_{12}(\mathbf{b}\cdot\nabla\times\mathbf{c})^2 + \frac{1}{2}A_{21}(\mathbf{c}\cdot\nabla\times\mathbf{b})^2 \\ & + A_{11}(\mathbf{b}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b}) \\ & + \frac{1}{2}B_1(\nabla\cdot\mathbf{b})^2 + \frac{1}{2}B_2(\nabla\cdot\mathbf{c})^2 \\ & + \frac{1}{2}B_3\left[\frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b} + \mathbf{c}\cdot\nabla\times\mathbf{c})\right]^2 \\ & + B_{13}(\nabla\cdot\mathbf{b})\left[\frac{1}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b} + \mathbf{c}\cdot\nabla\times\mathbf{c})\right] \\ & + C_1(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c}) + C_2(\nabla\cdot\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{b}), \end{aligned} \tag{10}$$

where the elastic constants A_i, B_i and C_i are related to those introduced by the Orsay Group [6], the slight difference being that $A_{11} = -\frac{1}{2}A_{11}^{\mathrm{Orsay}}$ and $C_1 = -C_1^{\mathrm{Orsay}}$. A physical interpretation of these constants and their related deformations can be found in [1].

It is known that the elastic constants obey the following inequalities [1]:

$$A_{12}, A_{21}, B_1, B_2, B_3 > 0, \tag{11}$$

$$A_{12}A_{21} - A_{11}^2 > 0, \tag{12}$$

$$B_1B_3 - B_{13}^2 > 0, \tag{13}$$

$$B_2A_{12} - C_1^2 > 0, \tag{14}$$

$$B_2A_{21} - C_2^2 > 0. \tag{15}$$

We now derive some new inequalities which will be useful later. Clearly

$$(A_{12} \pm A_{11})^2 = A_{12}^2 + A_{11}^2 \pm 2A_{12}A_{11} > 0. \tag{16}$$

Adding expression (12) to these inequalities and dividing by A_{12}, which is positive, gives the two inequalities

$$A_{12} + A_{21} + 2A_{11} > 0, \qquad A_{12} + A_{21} - 2A_{11} > 0. \tag{17}$$

This also implies that

$$|A_{11}| < \frac{1}{2}(A_{12} + A_{21}). \tag{18}$$

Similarly, by considering the quantities $(B_1 \pm B_{13})^2$, $(B_2 \pm C_1)^2$, $(B_2 \pm C_2)^2$ and adding the inequalities (13), (14), (15) to each quantity, respectively (and suitably dividing by either B_1 or B_2 when required) we arrive at the inequalities

$$B_1 + B_3 \pm 2B_{13} > 0 \Rightarrow |B_{13}| < \frac{1}{2}(B_1 + B_3), \tag{19}$$

$$B_2 + A_{12} \pm 2C_1 > 0 \Rightarrow |C_1| < \frac{1}{2}(B_2 + A_{12}), \tag{20}$$

$$B_2 + A_{21} \pm 2C_2 > 0 \Rightarrow |C_2| < \frac{1}{2}(B_2 + A_{21}). \tag{21}$$

The electric energy density w_e is given by [4, p. 287]

$$w_e = -\frac{1}{2}\varepsilon_a\varepsilon_0(\mathbf{n}\cdot\mathbf{E})^2, \tag{22}$$

where ε_a is the dielectric anisotropy of the liquid crystal and ε_0 is the permittivity of free space. It is assumed throughout that $\varepsilon_a > 0$.

The calculation of the equilibrium configuration is obtained by minimizing the total energy integral W over a volume Ω given in cylindrical coordinates by

$$W = \int_\Omega (w_b + w_e) r\,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}z = \int_\Omega \bar{w}\,\mathrm{d}r\,\mathrm{d}\alpha\,\mathrm{d}z, \tag{23}$$

putting $\bar{w} = (w_b + w_e)r$. The resulting Euler–Lagrange equation is derived from

$$\frac{\partial}{\partial r}\frac{\partial\bar{w}}{\partial(\phi_{,r})} + \frac{\partial}{\partial\alpha}\frac{\partial\bar{w}}{\partial(\phi_{,\alpha})} - \frac{\partial\bar{w}}{\partial\phi} = 0, \tag{24}$$

where a comma denotes partial differentiation with respect to the variable it precedes. Working in cylindrical coordinates, we substitute equations (5), (6) into (10) and equations (7), (8) into (22) to find

$$\begin{aligned} \bar{w} = & \frac{1}{2r}(A_{12}\sin^4\phi + A_{21}\cos^4\phi - 2A_{11}\sin^2\phi\cos^2\phi) \\ & + \frac{1}{2r}(B_1\sin^2\phi + B_2\cos^2\phi)(\phi_{,\alpha})^2 \\ & + \frac{r}{2}B_3(\phi_{,r})^2 + B_{13}\sin\phi\,\phi_{,\alpha}\phi_{,r} \\ & + \frac{1}{r}(C_1\sin^2\phi - C_2\cos^2\phi)\cos\phi\,\phi_{,\alpha} \\ & - \frac{1}{2}\varepsilon_a\varepsilon_0\frac{U^2}{r\beta^2}\sin^2\theta\sin^2\phi. \end{aligned} \tag{25}$$

Substituting equation (25) into (24) and multiplying throughout by r yields the following governing equilibrium equation for the wedge problem

$$(B_1 \sin^2\phi + B_2\cos^2\phi)\phi_{,\alpha\alpha} + \frac{1}{2}(B_1 - B_2)\sin 2\phi(\phi_{,\alpha})^2$$
$$+ rB_{13}\cos\phi\,\phi_{,\alpha}\phi_{,r} + 2rB_{13}\sin\phi\,\phi_{,r\alpha} + rB_3\phi_{,r} + r^2B_3\phi_{,rr}$$
$$+ 2[(A_{21}+A_{11})\cos^2\phi - (A_{12}+A_{11})\sin^2\phi]\sin\phi\cos\phi$$
$$+ \varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2 \sin^2\theta\sin\phi\cos\phi = 0. \qquad (26)$$

For the magnetic case w_e is replaced by w_m defined by [4]

$$w_m = -\frac{\chi_a}{2\mu_0}(\mathbf{n}\cdot\mathbf{B})^2, \qquad (27)$$

where μ_0 is the permeability of free space and χ_a is the magnetic anisotropy of the liquid crystal. It is assumed that $\chi_a > 0$. We are now in a position to seek solutions of equation (26) and its magnetic counterpart subject to suitable boundary conditions.

Remark: When there is no field present, three constant stable solutions are possible (see Carlsson *et al.* [1]) depending upon the signs of the A_i constants. If this stable solution is different from the imposed boundary conditions, there is the possibility of a non-zero variable solution for ϕ. Such variable solutions can occur in the wedge geometry when the wedge angle is greater than a critical angle. This is discussed in §4 after equation (52). An analogous situation arises in the concentric cylinder case and a critical radius ratio is obtained at the end of §6.

4. α-dependent solution in a wedge

Restricting attention to the case when $\phi = \phi(\alpha)$ equation (26) reduces to

$$(B_1\sin^2\phi + B_2\cos^2\phi)\phi'' + \frac{1}{2}(B_1 - B_2)\sin(2\phi)(\phi')^2$$
$$- 2(A_{12}+A_{21}+2A_{11})\sin^3\phi\cos\phi + \delta^2\sin\phi\cos\phi = 0, \qquad (28)$$

where a prime denotes differentiation with respect to α and for convenience we set

$$\delta^2 = \varepsilon_a\varepsilon_0(U/\beta)^2\sin^2\theta + 2(A_{21}+A_{11}). \qquad (29)$$

Multiplying equation (28) by ϕ' allows it to be reformulated as

$$\frac{d}{d\alpha}[(\phi')^2(B_1\sin^2\phi + B_2\cos^2\phi)$$
$$- (A_{12}+A_{21}+2A_{11})\sin^4\phi + \delta^2\sin^2\phi] = 0. \qquad (30)$$

In conjunction with the boundary conditions

$$\phi(0) = \phi(\beta) = 0 \qquad (31)$$

we assume that the distortion is symmetric about $\alpha = \beta/2$, that is,

$$\phi(\alpha) = \phi(\beta - \alpha), \qquad 0 \leqslant \alpha \leqslant \beta/2, \qquad (32)$$
$$\phi'(\beta/2) = 0, \qquad \phi(\beta/2) = \phi_m, \qquad (33)$$

where, without loss of generality, we suppose that $\phi_m > 0$. Upon integration, taking into account condition (33), equation (30) gives

$$(B_1\sin^2\phi + B_2\cos^2\phi)(\phi')^2$$
$$= \delta^2(\sin^2\phi_m - \sin^2\phi)$$
$$- (A_{12}+A_{21}+2A_{11})(\sin^4\phi_m - \sin^4\phi). \qquad (34)$$

Since $B_1 > 0$, $B_2 > 0$ for real solutions of this type to exist, the right-hand side of equation (34) must be positive for all ϕ and ϕ_m between 0 and $\pi/2$. In view of inequality (17) it is necessary to assume that

$$\delta^2 - 2(A_{12}+A_{21}+2A_{11}) > 0 \qquad (35)$$

for a solution of the form of equation (6) with $\phi = \phi(\alpha)$ to exist. A further integration of equation (34) then gives

$$\alpha = \int_0^{\phi} \frac{(B_1\sin^2\xi + B_2\cos^2\xi)^{1/2}}{[F(\xi,\phi_m)(\sin^2\phi_m - \sin^2\xi)]^{1/2}}\,d\xi, \qquad 0 \leqslant \alpha \leqslant \beta/2, \qquad (36)$$

where

$$F(\xi,\phi_m) = \delta^2 - (A_{12}+A_{21}+2A_{11})(\sin^2\phi_m + \sin^2\xi). \qquad (37)$$

The solution is completed by expression (32) and the parameter ϕ_m, the angle β and the applied voltage U must satisfy

$$\frac{\beta}{2} = \int_0^{\phi_m} \frac{(B_1\sin^2\xi + B_2\cos^2\xi)^{1/2}}{[F(\xi,\phi_m)(\sin^2\phi_m - \sin^2\xi)]^{1/2}}\,d\xi. \qquad (38)$$

ϕ_m therefore depends upon β, U and the material constants. For a given liquid crystal, if we fix β then a variation in U gives rise to a variation in ϕ_m.

Making the substitution

$$\sin\xi = \sin\phi_m\sin\nu \qquad (39)$$

equation (38) becomes

$$\frac{\beta}{2} = \int_0^{\pi/2} \{(B_1 - B_2)\sin^2\phi_m\sin^2\nu + B_2\}^{1/2}$$
$$\times\{(1 - \sin^2\phi_m\sin^2\nu)[\delta^2 - (A_{12}+A_{21}+2A_{11})$$
$$\times(1+\sin^2\nu)\sin^2\phi_m]\}^{-1/2}\,d\nu. \qquad (40)$$

Since this integrand is a continuous function of ϕ_m and ν for $\phi_m \in (-\pi/2, \pi/2)$ and $\nu \in [0, \pi/2)$, it follows that the integral in equation (40) is a continuous function of ϕ_m for $\phi_m \in (-\pi/2, \pi/2)$. Also, for fixed U/β, the integrand is an even function of ϕ_m and so we need only consider the behaviour for $\phi_m \in [0, \pi/2)$. Since $A_{12}+A_{21}+2A_{11} > 0$, it can be shown that for such values of ϕ_m the integral is a monotonic increasing function of ϕ_m taking all values greater than its minimum which occurs at $\phi_m=0$. Taking the limit as $\phi_m \to 0$ in equation (40), the minimum value of the right-hand side is $\pi B_2^{1/2}/(2\delta)$ and so the solution (6) can only exist if the voltage U and the angle β are such that

$$\delta > \pi B_2^{1/2}/\beta \tag{41}$$

in which case both this solution and the uniform orientation $\phi \equiv 0$ are possible solutions. Squaring inequality (41) shows that for a given β there exists a critical voltage U_c given by

$$\varepsilon_a \varepsilon_0 U_c^2 \sin^2\theta = \pi^2 B_2 - 2\beta^2 (A_{21} + A_{11}). \tag{42}$$

It is known [1] that for small θ

$$A_{21} + A_{11} = (\bar{A}_{21} + \bar{A}_{11})\theta^2, \tag{43}$$

$$B_2 = \bar{B}_2 \theta^2, \qquad B_3 = \bar{B}_3 \theta^2, \qquad B_{13} = \bar{B}_{13}\theta^3, \tag{44}$$

where the constants $\bar{A}_i$, $\bar{B}_i$ can be assumed to be only weakly temperature dependent (that is, independent of θ). Using these relationships in the case of small θ reduces the threshold (42) to the critical value derived by Carlsson *et al.* [1, equation (31)], namely

$$\varepsilon_a \varepsilon_0 U_c^2 = \pi^2 \bar{B}_2 - 2\beta^2 (\bar{A}_{21} + \bar{A}_{11}). \tag{45}$$

To examine which of the two solutions is more likely to occur when expression (41) holds, we examine the difference between the energies of the solution $\phi \equiv 0$ and the non-linear solution given by equations (32), (33), (36) and (38). This comparison determines which solution is energetically more favourable, the solution with the lower energy being preferred. Using equation (23), the total energy for the sample occupying the region $a \leqslant r \leqslant b$, $0 \leqslant \alpha \leqslant \beta$, $0 \leqslant z \leqslant h$ is given by

$$W = \int_a^b \mathrm{d}r \int_0^\beta \mathrm{d}\alpha \int_0^h \bar{w}\, \mathrm{d}z. \tag{46}$$

Putting

$$l = \ln\left(\frac{b}{a}\right), \tag{47}$$

setting $\Delta W = W(\phi(\alpha)) - W(\phi \equiv 0)$ and using equation (25) we obtain

$$\begin{aligned}\Delta W &= \frac{1}{2} hl \int_0^\beta \Bigg\{ (B_1 \sin^2\phi + B_2 \cos^2\phi)(\phi')^2 \\ &\quad + (A_{12} + A_{21} + 2A_{11}) \sin^4\phi - \delta^2 \sin^2\phi \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{3}(C_1 + C_2)\sin^3\phi - C_2 \sin\phi\right]\Bigg\} \mathrm{d}\alpha \\ &= \frac{1}{2} hl \int_0^\beta \{(B_1 \sin^2\phi + B_2\cos^2\phi)(\phi')^2 \\ &\quad + (A_{12} + A_{21} + 2A_{11})\sin^4\phi - \delta^2 \sin^2\phi\}\, \mathrm{d}\alpha \end{aligned} \tag{48}$$

using the boundary conditions (31). Using equations (34), (37) and (39) this becomes

$$\begin{aligned}\Delta W &= hl \int_0^{\beta/2} [\delta^2(\sin^2\phi_m - 2\sin^2\phi) \\ &\quad - (A_{12} + A_{21} + 2A_{11})(\sin^4\phi_m - 2\sin^4\phi)]\, \mathrm{d}\alpha \\ &= hl \int_0^{\phi_m} \frac{(B_1 \sin^2\phi + B_2\cos^2\phi)^{1/2}}{[F(\phi,\phi_m)(\sin^2\phi_m - \sin^2\phi)]^{1/2}} \\ &\quad \times [\delta^2(\sin^2\phi_m - 2\sin^2\phi) \\ &\quad - (A_{12} + A_{21} + 2A_{11})(\sin^4\phi_m - 2\sin^4\phi)]\, \mathrm{d}\phi \end{aligned} \tag{49}$$

$$\begin{aligned} &= hl \sin^2\phi_m \int_0^{\pi/2} \frac{(B_1 \sin^2\phi + B_2 \cos^2\phi)^{1/2}}{[F(\phi,\phi_m)(1 - \sin^2\phi_m \sin^2\nu)]^{1/2}} \\ &\quad \times [\cos(2\nu) F(\phi,\phi_m) \\ &\quad - (A_{12} + A_{21} + 2A_{11}) \sin^2\phi_m \sin^2\nu]\, \mathrm{d}\nu \\ &= -\frac{1}{2} hl \sin^2\phi_m \int_0^{\pi/2} \sin(2\nu) \\ &\quad \frac{\mathrm{d}}{\mathrm{d}\nu} \times \left\{\frac{[F(\phi,\phi_m)(B_1\sin^2\phi + B_2\cos^2\phi)]^{1/2}}{\cos\phi}\right\} \mathrm{d}\nu \\ &\quad - hl(A_{12} + A_{21} + 2A_{11})\sin^4\phi_m \\ &\quad \times \int_0^{\pi/2} \left[\frac{B_1 \sin^2\phi + B_2\cos^2\phi}{F(\phi,\phi_m)(1 - \sin^2\phi_m \sin^2\nu)}\right]^{1/2} \sin^2\nu\, \mathrm{d}\nu \end{aligned} \tag{50}$$

using integration by parts. Clearly, in view of the inequalities (17), the second term in the last equality above is negative. Straightforward differentiation also shows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\nu}\left\{\frac{[F(\phi,\phi_m)(B_1 \sin^2\phi + B_2\cos^2\phi)]^{1/2}}{\cos\phi}\right\} \\ &= \frac{B_1 \sin(2\nu)\sin^2\phi_m}{2\cos^3\phi [F(\phi,\phi_m)(B_1\sin^2\phi + B_2\cos^2\phi)]^{1/2}} \\ &\quad \times \{\delta^2 - (A_{12} + A_{21} + 2A_{11}) \\ &\quad \times [\sin^2\phi_m + 1 + (B_2/B_1 - 1)\cos^4\phi]\} \\ &> 0 \end{aligned} \tag{51}$$

for all $\phi, \phi_m \in [0, \pi/2)$ and for $0 < \nu < \pi/2$ under assumption (35) if $B_2 \leqslant B_1$. It is also positive if $B_2 > B_1$ provided $\sin^2\phi_m + B_2/B_1 < 2$ which is only possible for $B_1 < B_2 < 2B_1$. It therefore follows that if $B_2 \leqslant B_1$ or $B_1 < B_2 < 2B_1$ then

$$\Delta W < 0 \tag{52}$$

and so we anticipate the variable solution to occur in preference to the uniform solution $\phi \equiv 0$ whenever U and β satisfy equation (41). (If ϕ_m is assumed to be sufficiently small, then the inequalities involving B_1 and

B_2 need not be considered since B_2 would then be the dominant constant.)

It is worth commenting further on expressions (35) and (42). For the variable non-linear solution to exist, it is necessary for (35) to hold and for the right-hand side of (42) to be positive. We also need the right-hand side of equation (29) to be positive for all $U > U_c$, but this follows from (29) and (42) since $B_2 > 0$. The conditions for these restrictions to be satisfied depend upon the sign of the combinations $A_{12} + A_{11}$ and $A_{21} + A_{11}$. Since $A_{12} + A_{21} + 2A_{11} > 0$, both combinations cannot be negative. If $A_{12} + A_{11} < 0$ then $A_{21} + A_{11} > 0$, (35) holds and we need

$$\pi^2 B_2 - 2\beta^2 (A_{21} + A_{11}) > 0, \tag{53}$$

or

$$\beta < \frac{\pi B_2^{1/2}}{[2(A_{21} + A_{11})]^{1/2}}. \tag{54}$$

This leads to a critical wedge angle β_c given by

$$\beta_c = \frac{\pi B_2^{1/2}}{[2(A_{21} + A_{11})]^{1/2}}. \tag{55}$$

If $A_{12} + A_{11} > 0$, then $A_{21} + A_{11}$ can be either positive or negative. Both these cases can be considered together. For expression (35) to be satisfied for all $U \geq U_c$ we need

$$\pi^2 B_2 - 2\beta^2 (A_{12} + A_{21} + 2A_{11}) \geq 0 \tag{56}$$

and when this inequality holds, the right-hand side of equation (42) is positive irrespective of the sign of $A_{21} + A_{11}$. This leads to a second critical angle

$$\bar{\beta}_c = \frac{\pi B_2^{1/2}}{[2(A_{12} + A_{21} + 2A_{11})]^{1/2}}. \tag{57}$$

As mentioned at the end of § 3, when $\mathbf{E} \equiv \mathbf{0}$ $\phi \equiv 0$ need not be the only solution to equation (28) satisfying the zero boundary conditions (31). If $A_{21} + A_{11} > 0$ and there is no field, then $\delta^2 = 2(A_{21} + A_{11}) > 0$ and, to satisfy expression (35), we must have $A_{12} + A_{11} < 0$. In this case, when $A_{21} + A_{11} > 0$, $A_{12} + A_{11} < 0$, the stable constant solution is $\phi = \pi/2$ and so there is likely to be competition with the boundary condition $\phi = 0$ for a large enough wedge angle, giving rise to a variable solution. From the above working a non-zero symmetric variable solution exists whenever inequality (41) is true, that is, whenever

$$\beta > \beta_c, \tag{58}$$

where β_c is given by equation (55). When $0 \leq \beta \leq \beta_c$, $\phi \equiv 0$ is the only symmetric solution satisfying the zero boundary conditions. For $\beta > \beta_c$ both the zero solution and the variable solution $\phi(\alpha)$ (given in equation (36) with $E = 0$) are possible and an energy argument almost the same as that given above verifies that expression (52) holds for the energy difference between these two solutions. Hence, in these circumstances the system prefers the variable solution when there is no field present and $\beta > \beta_c$. The solution for $\phi(s)$ can be obtained numerically if necessary from equation (36) once ϕ_m has been calculated from equation (38), where suitable estimates for the elastic constants have been inserted as needed. Clearly, from expression (40), since β must be physically restricted between 0 and 2π, there must be a maximum allowed value for ϕ_m which can be calculated numerically if desired. The critical parameter in equation (55) is the same as that discussed in [1, equation (35)] with the advantage here being that the solution for $\phi(\alpha)$ when $\beta > \beta_c$ can be obtained.

5. *r*- and α-dependent solutions in a wedge

For a finite wedge it is necessary to allow ϕ to vary with both r and α. Whilst equation (26) cannot be solved analytically, it is possible to obtain the Fréedericksz threshold and gain some useful information about the constant B_3 by considering a linearized problem.

Linearizing equation (26) in ϕ results in the equation

$$B_2 \phi_{,\alpha\alpha} + B_3 \phi_{,ss} + \delta^2 \phi = 0, \tag{59}$$

where the new variable s is defined by

$$s = \ln\left(\frac{r}{a}\right) \tag{60}$$

and δ^2 is given by equation (29). We now consider a sample in the finite region $0 \leq s \leq l$, $0 \leq \alpha \leq \beta$, where l is given by definition (47), subject to $\phi(s, \alpha)$ being zero on the boundaries at $s = 0, l$ and $\alpha = 0, \beta$. Using definitions (47) and (60) this gives

$$\phi(0, \alpha) = \phi(l, \alpha) = 0, \qquad 0 \leq \alpha \leq \beta, \tag{61}$$

$$\phi(s, 0) = \phi(s, \beta) = 0, \qquad 0 \leq s \leq l. \tag{62}$$

Equation (59) is a form of the Helmholtz equation and can be solved using the standard technique of separation of variables. A simple ansatz corresponding to the first eigenfunction of such a solution satisfying the boundary conditions is

$$\phi(s, \alpha) = \phi_m \sin\left(\frac{\pi}{l} s\right) \sin\left(\frac{\pi}{\beta} \alpha\right) \tag{63}$$

for some suitably small constant ϕ_m, this perturbation reaching its maximum when $s = l/2$ and $\alpha = \beta/2$. Substituting equation (63) into (59) gives the critical threshold voltage U_c as

$$\varepsilon_a \varepsilon_0 U_c^2 \sin^2 \theta = \pi^2 B_2 - 2\beta^2 (A_{21} + A_{11}) + \left(\frac{\pi}{l}\beta\right)^2 B_3. \tag{64}$$

Clearly, this threshold is approximated to by the threshold given in equation (42) in a radially large sample when l is large

An alternative and more direct derivation of the Fréedericksz threshold is possible using the energy. Equation (63) is the simplest perturbation that can be made that satisfies the boundary conditions when ϕ_m is sufficiently small. An energy comparison between this perturbation and the initial unperturbed configuration $\phi \equiv 0$ can be carried out by inserting (63) directly into the energy given by equations (23) and (25) and retaining terms up to the order ϕ_m^2, assuming the sample Ω is of thickness h in the z-direction. Setting

$$\Delta W = W(\phi(r, \alpha)) - W(\phi \equiv 0)$$

we obtain

$$\Delta W = h \int_0^l \int_0^\beta \left\{ -(A_{21} + A_{11})\phi^2 - \frac{1}{2}\varepsilon_a \varepsilon_0 \left(\frac{U}{\beta}\right)^2 \sin^2\theta\phi^2 + \frac{1}{2}[B_2(\phi_{,\alpha})^2 - 2C_2\phi_{,\alpha}] + \frac{1}{2}B_3(\phi_{,s})^2 \right\} \mathrm{d}s\,\mathrm{d}\alpha. \quad (65)$$

From the boundary conditions (62)

$$\int_0^\beta \phi_{,\alpha}\mathrm{d}\alpha = 0 \quad (66)$$

and hence equation (65) is

$$\begin{aligned}\Delta W = {} & h\frac{\phi_m^2}{2}\int_0^l \sin^2\left(\frac{\pi}{l}s\right)\mathrm{d}s \int_0^\beta \\ & \times \left\{\left[-2(A_{21}+A_{11}) - \varepsilon_a\varepsilon_0\left(\frac{U}{\beta}\right)^2 \sin^2\theta\right] \times \sin^2\left(\frac{\pi}{\beta}\alpha\right) + B_2\left(\frac{\pi}{\beta}\right)^2\cos^2\left(\frac{\pi}{\beta}\alpha\right)\right\}\mathrm{d}\alpha \\ & + h\frac{\phi_m^2}{2}\int_0^l \cos^2\left(\frac{\pi}{l}s\right)\mathrm{d}s\int_0^\beta B_3\left(\frac{\pi}{l}\right)^2 \sin^2\left(\frac{\pi}{\beta}\alpha\right)\mathrm{d}\alpha \\ = {} & hl\phi_m^2\frac{\beta}{8}\left[-2(A_{21}+A_{11}) - \varepsilon_a\varepsilon_0 \sin^2\theta\left(\frac{U}{\beta}\right)^2 + B_2\left(\frac{\pi}{\beta}\right)^2 + B_3\left(\frac{\pi}{l}\right)^2\right]. \quad (67)\end{aligned}$$

Clearly,

$$\Delta W < 0 \qquad \text{only if} \quad U > U_c, \quad (68)$$

where U_c is the threshold given by equation (64) and therefore this perturbed solution $\phi(s, \alpha)$ is energetically favoured near the critical threshold.

It is perhaps worth remarking that if there are no boundary conditions on the outer radial boundary then

$$\phi(s, \alpha) = \phi_m s \sin\left(\frac{\pi}{\beta}\alpha\right) \quad (69)$$

is always a form of solution to equation (59) which, when inserted into (59), gives exactly the same threshold as the α-dependent solution given in equation (42). Unfortunately this solution may grow unacceptably large for the linearized version of the equations to be valid.

6. *r*-Dependent solution between cylinders

So far the solution (4) to (6) has been applied to the case when an electric field of the form (8) is applied to a sample of smectic liquid crystal occupying a wedge. In this section, a magnetic field of the form (9) is applied to a sample between two coaxial circular cylinders of radii a and b ($a < b$), the z-axis coinciding with the common axis of the cylinders. We again take $\phi = 0$ on the boundaries so that

$$\phi(a) = 0, \qquad \phi(b) = 0. \quad (70)$$

These boundary conditions correspond to $\mathbf{n} = \mathbf{n}_b$ where

$$\mathbf{n}_b = \hat{\mathbf{r}}\cos\theta + \hat{\mathbf{z}}\sin\theta, \quad (71)$$

that is, on the boundary the director is in the rz-plane making an angle $\pi/2 - \theta$ with the cylinders. The corresponding equation for ϕ when a magnetic field is applied is obtained from equation (26) by replacing $\varepsilon_a\varepsilon_0(U/\beta)^2$ with $\chi_a B^2 a^2/\mu_0$, these quantities being defined earlier. Taking $\phi = \phi(r)$, equation (26) reduces to

$$B_3[r^2\phi''(r) + r\phi'(r)] - 2(A_{12} + A_{21} + 2A_{11})\sin^3\phi\cos\phi + \gamma^2 \sin\phi\cos\phi = 0, \quad (72)$$

where

$$\gamma^2 = \frac{\chi_a}{\mu_0}B^2a^2\sin^2\theta + 2(A_{21} + A_{11}). \quad (73)$$

When there is no magnetic field there is a solution of equation (72) satisfying the boundary conditions (70), namely,

$$\phi \equiv 0. \quad (74)$$

Although (74) remains a solution when $\mathbf{B}$ is non-zero, there are other possibilities. Introducing the variable s and the constant l defined in (47) and (60), equation (72) becomes

$$B_3\phi''(s) - 2(A_{12} + A_{21} + 2A_{11})\sin^3\phi\cos\phi + \gamma^2\sin\phi\cos\phi = 0 \quad (75)$$

with

$$\phi(0) = 0, \qquad \phi(l) = 0. \quad (76)$$

Multiplying equation (75) by $\phi'(s)$, it follows that

$$\frac{d}{ds}[B_3(\phi')^2 - (A_{12} + A_{21} + 2A_{11})\sin^4\phi + \gamma^2\sin^2\phi] = 0. \tag{77}$$

In view of the boundary conditions (76) it is natural to look for a solution in which $\phi(s)$ is symmetric about $s = l/2$ so that

$$\phi(s) = \phi(l - s), \qquad 0 \leqslant s \leqslant l/2, \tag{78}$$

and

$$\phi'(l/2) = 0, \qquad \phi(l/2) = \phi_m, \tag{79}$$

where, without loss of generality, we suppose that $\phi_m > 0$. Integrating equation (77), taking into account (79), gives

$$B_3[\phi'(s)]^2 = \gamma^2(\sin^2\phi_m - \sin^2\phi) - (A_{12} + A_{21} + 2A_{11})(\sin^4\phi_m - \sin^4\phi). \tag{80}$$

Since $B_3 > 0$ for real solutions of this type to exist, we require that the right-hand side of equation (80) is positive for all ϕ and ϕ_m between 0 and $\pi/2$. Condition (35) with δ replaced by γ ensures this. A further integration of equation (80) then gives

$$s = B_3^{1/2}\int_0^{\phi}[F(\xi, \phi_m)(\sin^2\phi_m - \sin^2\xi)]^{-1/2}\,d\xi, \qquad 0 \leqslant s \leqslant l/2, \tag{81}$$

where here $F(\xi, \phi_m)$ is the function defined in equation (37) with δ replaced by γ. The solution is completed by equation (78) and the parameter ϕ_m, the gap width and the magnetic field must satisfy

$$l = 2B_3^{1/2}\int_0^{\phi_m}[F(\xi, \phi_m)(\sin^2\phi_m - \sin^2\xi)]^{-1/2}\,d\xi. \tag{82}$$

Using the variable v defined by (39), this becomes

$$l = 2B_3^{1/2}\int_0^{\pi/2}(1 - \sin^2\phi_m\sin^2 v)^{-1/2} \times [\gamma^2 - (A_{12} + A_{21} + 2A_{11})(1 + \sin^2 v)\sin^2\phi_m]^{-1/2}\,dv. \tag{83}$$

Repeating the argument following equation (40), it follows that l is a monotonic increasing function of ϕ_m taking all values greater than its minimum which occurs at $\phi_m = 0$. The minimum value of the right-hand side of equation (83) is $\pi B_3^{1/2}/\gamma$ and so the solution (6) can only exist if the magnetic field strength B and the ratio of the radii a and b are such that

$$\gamma > \pi B_3^{1/2}/l, \tag{84}$$

in which case both this solution and the solution $\phi \equiv 0$ are possible. Squaring inequality (84) shows that for given radii there exists a critical field strength B_c given by

$$\chi_a\mu_0^{-1}B_c^2 a^2\sin^2\theta = \pi^2 B_3[\ln(b/a)]^{-2} - 2(A_{21} + A_{11}). \tag{85}$$

As before, we consider the difference in the energies associated with these solutions. In this case using equations (23), (25) and (46) (with $\beta = 2\pi$) and equation (80)

$$\begin{aligned}\Delta W &= \pi h\int_0^l\left[B_3\left(\frac{d\phi}{ds}\right)^2 + (A_{12} + A_{21} + 2A_{11})\sin^4\phi - \gamma^2\sin^2\phi\right]ds \\ &= 2\pi h\int_0^{l/2}[\gamma^2(\sin^2\phi_m - 2\sin^2\phi) - (A_{12} + A_{21} + 2A_{11})(\sin^4\phi_m - 2\sin^4\phi)]\,ds \\ &= 2\pi h B_3^{1/2}\int_0^{\phi_m}[F(\phi, \phi_m)(\sin^2\phi_m - \sin^2\phi)]^{-1/2} \\ &\quad \times[\gamma^2(\sin^2\phi_m - 2\sin^2\phi) - (A_{12} + A_{21} + 2A_{11})(\sin^4\phi_m - 2\sin^4\phi)]\,d\phi.\end{aligned} \tag{86}$$

This integral is a special case of the integral in equation (49) and by putting $B_1 = B_2 = B_3$ in the working presented there, it follows that

$$\Delta W < 0 \tag{87}$$

in this case and so we anticipate the variable solution to occur in preference to the uniform solution $\phi \equiv 0$ whenever B, a and b satisfy inequality (84).

For the above variable solution to exist, the condition (35) with δ replaced by γ has to be satisfied and the right-hand side of equation (85) has to be positive. Following a similar argument to that given in §4 for the wedge, the corresponding conditions follow from equations (53), (54) and (56) by replacing β by $\ln(b/a)$ and B_2 by B_3. In particular in the case when $A_{12} + A_{11} < 0$, $A_{21} + A_{11} > 0$ there is a critical radius ratio l_c given by

$$l_c = \frac{\pi B_3^{1/2}}{[2(A_{21} + A_{11})]^{1/2}}. \tag{88}$$

As in §4, there can be variable solutions when no field is present since when $\mathbf{B} \equiv \mathbf{0}$, $\phi \equiv 0$ need not be the only solution to equation (72) satisfying the zero boundary conditions. For example, when there is no field, if $A_{21} + A_{11} > 0$ then $\gamma^2 = 2(A_{21} + A_{11}) > 0$ and, if the corresponding condition in (35) with δ replaced by γ is satisfied, then we must have $A_{12} + A_{11} < 0$. When this is the case, there is a symmetric non-zero variable solution provided inequality (84) holds, that is,

$$l > l_c, \quad (89)$$

where l_c is given by equation (88). The solution $\phi \equiv 0$ is the only symmetric solution satisfying the zero boundary conditions for $0 \leqslant l \leqslant l_c$ while both this zero solution and the symmetric variable solution $\phi(s)$ given by equation (81) exist for $l > l_c$. When $l > l_c$ an almost identical argument to that used above shows that $\Delta W < 0$ for the energy difference between these solutions, which indicates the system's preference for the variable solution when $l > l_c$. The behaviour of the solution $\phi(s)$ in these circumstances can be calculated numerically via equation (81) if desired, once estimates for the elastic constants are inserted and ϕ_m has been calculated from equation (82). It is clear from equation (83) that $\phi_m \to \pi/2$ as $l \to \infty$.

7. Discussion

The results presented in this article generalize those in [1] to the full non-linear differential equation (26). With the aid of some new inequalities given in (17) and (18), a full non-linear analysis has been accomplished for the α-dependent smectic C wedge problem where no approximations were made in the smectic tilt angle θ. The non-linearly derived critical electric field strength U_c given by equation (42) was shown to reduce to the approximating threshold given in equation (45) which was previously derived in [1] by linearizing in the tilt angle θ. Introducing restrictions on the elastic constants and the wedge angle β (equations (53) or (56)), a non-linear solution (not identically zero) exists for $U > U_c$ and, provided $B_2 \leqslant B_1$ or $B_1 < B_2 \leqslant 2B_1$, this solution is energetically favoured over the zero solution for $U > U_c$. Clearly, since $B_2 > 0$, the wedge angle β can always be made sufficiently small so that the inequalities given in (53) and (56) are achieved. It would be of interest to know how restrictive these inequalities may actually be for experiments.

The dependence of the solution ϕ on the coordinates r and α in the wedge problem was discussed in §5 where equation (26) was linearized in ϕ while no approximations on θ were introduced. The resulting critical electric field threshold U_c is given by equation (64) which, unlike the threshold given in (42), shows the dependence of U_c upon the elastic constant B_3. For radially large samples, the B_3 contribution to U_c is then seen to become correspondingly smaller. A simple energy comparison (to second order in ϕ) of this solution and the $\phi \equiv 0$ solution further justified U_c as the critical threshold. The general conclusion is that introducing both α- and r-dependence into the solution ϕ does not greatly alter the critical threshold for radially large samples in a wedge geometry. Consequently, the assumption $\phi = \phi(\alpha)$ is well founded for the analysis carried out in §4 above.

One of the major assumptions throughout §4 and §5 is that the dielectric anisotropy ε_a is positive, which is generally known to be true for some smectic C liquid crystals. Nevertheless, there are also smectics which have $\varepsilon_a < 0$. This change in sign radically affects the governing equation (26): for example, the assumption (35) cannot be true for δ defined by (29) when U is very large because of inequality (17).

An r-dependent solution was found for equation (26) when a sample of smectic C liquid crystal is confined between two coaxial circular cylinders. The governing equation (72) with boundary conditions (70) was conveniently transformed to equation (80), making no approximations on the smectic tilt angle θ. A full non-linear analysis of equation (80) enabled the derivation of the critical magnetic field strength B_c, given by (85). This threshold has a form reminiscent of the electric field case given in (42), except that B_3 takes the place of B_2. From (85) it is seen that by varying the relative distance between the two coaxial cylinders it would be possible to measure B_3 and the combination $A_{21} + A_{11}$ from the same type of experiment; this is analogous to varying the wedge angle β in (42) to measure B_2 and $A_{21} + A_{11}$. An energy comparison between the $\phi \equiv 0$ solution and the non-linear variable solution for $B > B_c$ revealed the system's preference for the variable solution to occur above the Fréedericksz threshold. Conditions (53) and (56) for the wedge and the corresponding conditions for the cylinder may impose restrictions on the geometry. It should be mentioned that Fréedericksz transitions in planar layers of smectic were examined by Schiller and Pelzl [16] who obtained some experimental values for $B = B_1 \cos^2 \theta + B_3 \sin^2 \theta$. A general smectic elastic constant B has also been measured for various smectic materials at different temperatures by Pelzl *et al.* [17].

It was also demonstrated in §4 and §6 that non-linear variable solutions are possible for wedge and cylinder geometries in the absence of fields. If $A_{21} + A_{11} > 0$ and $A_{12} + A_{11} < 0$ then it was shown that there is a critical angle β_c for wedges and a critical radius ratio l_c for cylinders (given by equations (55) and (88), respectively) where values above these critical values can lead to non-zero solutions of the governing equilibrium equations which are energetically favourable. These solutions can be calculated numerically from equations (36) and (38) for the wedge and equations (81) and (82) for the cylindrical geometry using estimates for the elastic constants.

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